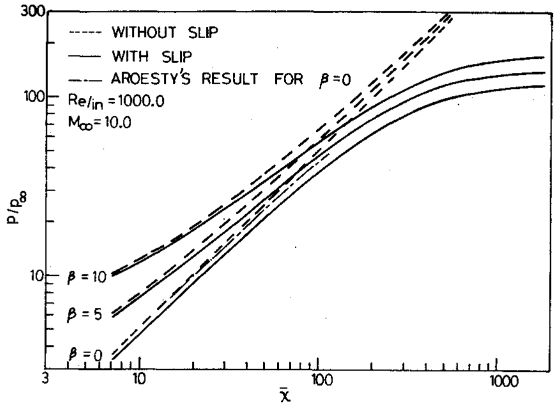


**Fig. 3 Variation of $p/p_\infty$ with $\bar{\chi}$.**

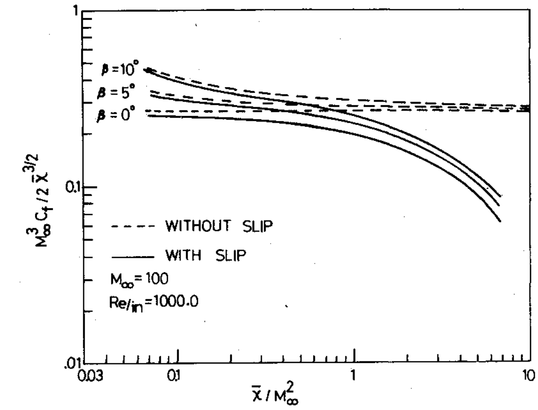


**Fig. 4 Variation of $M_\infty{}^3 c_f/2\bar{\chi}^{3/2}$ with $\bar{\chi}/M_\infty{}^2$.**

is increased. As expected, due to slip, Aroesty's result shows a slight decrease in pressure from the strong interaction value.

In Fig. 4, skin-friction coefficient is plotted against the rarefaction parameter $M_\infty (c/Re_x)^{1/2}$. It is seen again that slip reduces the skin friction. Also, increase in semiwedge angle increases skin friction. Sufficiently far downstream, both pressure and skin friction tend to their strong interaction value, respectively.


### References

[1] Shen, S. F., "An Estimate of Viscosity Effects on the Hypersonic Flow over an Insulated Wedge," *Journal of Mathematics and Physics*, Vol. 31, 1952, pp. 192–205.

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# Statics of Transversely Isotropic Beams

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### Introduction

RECENT papers[1–4] have demonstrated a few instances in which transverse isotropy plays a significant role in beam or plate analysis. These studies are potentially important since many of the newer materials considered for use in aerospace structural applications exhibit transverse isotropy. However, it is not generally realized that transverse isotropy plays a significant role in all kinds of static and dynamic analyses. Thus it is the purpose of this Note to provide a brief study of the effects of transverse isotropy on a variety of static beam problems.† The problems considered are the determination of Green's functions (influence functions), static deflections due to distributed loads, and beam-column deflections.‡ It will be shown that the effects of transverse isotropy can be quite dramatic, particularly when the beam boundary restraints are increased.

### Static Equations of Motion

Assuming displacements of the form $u(x, z) = z\psi(x)$ and $w(x, z) = w_0(x) + w(x)$, the needed strain-displacement relations are given by

$$\varepsilon_x = z d\psi/dx, \quad \varepsilon_{xz} = \tfrac{1}{2}(\psi + dw/dx) \tag{1}$$

Defining the in-plane modulus of elasticity to be $E$, the transverse shear modulus to be $G^*$, and Mindlin's shear correction factor to be $\kappa^2$ (which we will assume to be $\pi^2/12$), the stress-strain laws are given by

$$\sigma_x = Ez d\psi/dx, \quad \sigma_{xz} = \kappa^2 G^*(\psi + dw/dx) \tag{2}$$

Using (2), the shear and moment resultants are given by

$$Q_x = \kappa^2 G^* A^*(\psi + dw/dx) \tag{3}$$

$$M_x = -EI\, d\psi/dx \tag{4}$$

where $A^* = bh$, $I = bh^3/12$, $b$ is the beam width and $h$ is the beam depth. The sum of the vertical forces and $y$ axis moments are given by

$$q + dQ_x/dx + N_x d^2(w + w_0)/dx^2 = 0 \tag{5}$$

$$dM_x/dx + Q_x = 0 \tag{6}$$

where $q$ = transverse distributed loading; $w_0$ = initial (unstressed) displacement; $N_x$ = in-plane load. Putting (3) and (4) into (5) and (6) yields coupled equations for $w$ and $\psi$. That is,

$$\kappa^2 G^* A^*(d\psi/dx + d^2w/dx^2) + N_x d^2w/dx^2 = -N_x d^2w_0/dx^2 - q \tag{7}$$

$$-EI\, d^2\psi/dx^2 + \kappa^2 G^* A^*(\psi + dw/dx) = 0 \tag{8}$$

### Green's Functions

Rearranged forms of (3) and (4), that is

$$d\psi/dx = -M_x/EI \tag{9}$$

$$dw/dx = (Q_x/\kappa^2 G^* A^*) - \psi \tag{10}$$

are used to determine Green's functions for statically determinate beams in the following fashion: for a given $M_x$ and $Q_x$ (in the regions $0 \leqq x \leqq \zeta$ and $\zeta \leqq x \leqq l$), due to a specified unit generalized load (either force or moment), (9) may be integrated to find $\psi$ and then this result entered into (10) determines $w$ after another integration. The constants of integration are determined by the boundary conditions and by the matching conditions (slope and displacement) at the interface $x = \xi$.

For a cantilever beam we find that the complete set of Green's functions is given by,

$$\begin{aligned}
\overset{\delta F}{C}(x,\xi) &= \begin{cases} (x/6EI)[x(3\xi - x) + 6l^2S^*]\,; & x \leqq \xi \\ (\xi/6EI)[\xi(3x - \xi) + 6l^2S^*]\,; & x \geqq \xi \end{cases} \\
\overset{\delta M}{C}(x,\xi) &= \begin{cases} (x^2/2EI) & ; x \leqq \xi \\ (\xi/2EI)(2x - \xi) & ; x \geqq \xi \end{cases} \\
\overset{\psi F}{C}(x,\xi) &= \begin{cases} (x/2EI)(x - 2\xi) & ; x \leqq \xi \\ (-\xi^2/2EI) & ; x \geqq \xi \end{cases} \\
\overset{\psi M}{C}(x,\xi) &= \begin{cases} (-x/EI) & ; x \leqq \xi \\ (-\xi/EI) & ; x \geqq \xi \end{cases}
\end{aligned} \tag{11}$$

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† Dynamic problems will be considered in a sequel Note.

‡ Buckling problems have been treated previously[2,3] and thus will not be treated in this Note.

and for a simply supported beam we have the following results§

$$\frac{6EI}{l^3}\overset{\delta F}{C}(x,\xi) = \begin{cases} (x/l)(1-\xi/l)[(\xi/l)(2-\xi/l)-(x/l)^2+6S^*] \\ (\xi/l)(1-x/l)[(x/l)(2-x/l)-(\xi/l)^2+6S^*] \end{cases}$$

$$\frac{6EI}{l^2}\overset{\psi F}{C}(x,\xi) = \begin{cases} (1-\xi/l)[3(x/l)^2-(\xi/l)(2-\xi/l)] \\ (\xi/l)[6(x/l)-3(x/l)^2-(\xi/l)^2-2] \end{cases} \qquad (12)$$

For a clamped beam, we use the results of Eq.(11) to obtain the following

$$\overset{\delta F}{\bar{C}}(x,\xi) = \overset{\delta F}{C}(x,\xi) + \overset{\delta F}{C}(x,l)Q^* + \overset{\delta M}{C}(x,l)M^*$$
$$\overset{\psi F}{\bar{C}}(x,\xi) = \overset{\psi F}{C}(x,\xi) + \overset{\psi F}{C}(x,l)Q^* + \overset{\psi M}{C}(x,l)M^* \qquad (13)$$

and since $\overset{\delta F}{\bar{C}}(l,\xi) = \overset{\psi F}{\bar{C}}(l,\xi) = 0$ we have the following equations to solve for $Q^*$ and $M^*$

$$\overset{\delta F}{C}(l,l)Q^* + \overset{\delta M}{C}(l,l)M^* = -\overset{\delta F}{C}(l,\xi)$$
$$\overset{\psi F}{C}(l,l)Q^* + \overset{\psi M}{C}(l,l)M^* = -\overset{\psi F}{C}(l,\xi) \qquad (14)$$

which when re-inserted into Eq. (13) yields expressions for the clamped Green's functions.

Comparing the maximum non-dimensionalized deflections for some selected Green's functions it is seen that

$$(3EI/l^3)\overset{\delta F}{C}(l,l) = 1 + 3S^* \text{ (cantilever beam)} \qquad (15)$$

$$(48EI/l^3)C^{\delta F}(l/2,l/2) = 1 + 12S^* \text{ (simply supported beam)} \qquad (16)$$

$$(192EI/l^3)\bar{C}^{\delta F}(l/2,l/2) = 1 + 48S^* \text{ (clamped beam)} \qquad (17)$$

where $S^* = (E/G^*)(h/l)^2/\pi^2$. Noting that Eqs. (15–17) are unity for the classical beam case ($S^* = 0$) it is seen that when $S^* = 0.10$,† Eq. (15) is 1.30 times its classical value, Eq. (16) is 2.20 times its classical value, and Eq. (17) is 5.80 times its classical value. These results, as will be seen in the sequel, depict the general effects of transverse isotropy with regard to boundary conditions. That is, the more severe the end restraints, the *larger* the effect of transverse isotropy on deflection. Quite obviously this is a deleterious effect and has serious design implications.

### Static Solutions

For static problems with a distributed loading but no in-plane loads Eqs. (7) and (8) reduce to

$$d^2w/dx^2 + d\psi/dx = -q/\kappa^2G^*A^* \qquad (18)$$
$$-dw/dx + EI/\kappa^2G^*A^*\, d^2\psi/dx^2 - \psi = 0 \qquad (19)$$

Uncoupling these equations (e.g. by operator methods) the results are

$$d^4w/dx^4 = -(\kappa^2G^*A^*)^{-1}\, d^2q/dx^2 + q/EI \qquad (20)$$
$$d^4\psi/dx^4 = 1/EI\, dq/dx \qquad (21)$$

If the loading is constant, the $w$ and $\psi$ solutions for simple supports are given by

$$(24EI/q)w(x) = x[l^3 - 2lx^2 + x^3 + 12l^2S^*(l-x)] \qquad (22)$$
$$(24EI/q)\psi(x) = -[l^3 - 6lx^2 + 4x^3] \qquad (23)$$

and for a clamped beam with a constant loading $w$ and $\psi$ are given by

$$(24EI/q)w(x) = (l-x)x[(l-x)x + 12S^*l^2] \qquad (24)$$
$$(12EI/q)\psi(x) = -x(l-x)(l-2x) \qquad (25)$$

Comparing the maximum nondimensionalized deflections it is seen that

$$(384EI/5ql^4)w(l/2) = 1 + (48/5)S^* \text{ (simply supported beam)} \qquad (26)$$
$$(384EI/ql^4)w(l/2) = 1 + 48S^* \text{ (clamped beam)} \qquad (27)$$

Noting that Eqs. (26) and (27) are unity for the classical case ($S^* = 0$) it is seen that when $S^* = 0.10$, Eq. (26) is 1.96 times its classical value and Eq. (27) is 5.80 times its classical value. Again the effects of transverse isotropy are greatest when the beam is clamped.

### Beam Column

Letting $P = -N_x$ the appropriate forms of Eqs. (7) and (8) are

$$\kappa^2G^*A^*(d\psi/dx + d^2w/dx^2) - p(d^2/dx^2) + q = 0 \qquad (28)$$
$$-EI(d^2\psi/dx^2) + \kappa^2G^*A^*(\psi + dw/dx) = 0 \qquad (29)$$

and the uncoupled equations are given by

$$d^4\psi/dx^4 + k^2(d^2\psi/dx^2) = (k^2/p)(dq/dx)P \qquad (30)$$
$$d^4w/dx^4 + k^2(d^2w/dx^2) = (k^2/p)q - (k^2EI/P\kappa^2G^*A^*)(d^2q/dx^2) \qquad (31)$$

where

$$k^2 = \frac{P/EI}{(1 - P/\kappa^2G^*A^*)}$$

If the distributed loading is constant, $w$, for the simply supported case, is given by

$$w(x) = \frac{ql^4}{16EIu^4}\left[1 + \left(\frac{2u}{\pi}\right)^2 S\right]^2 \left[\frac{\cos(u - 2ux/l)}{\cos u} - 1 + \frac{2u^2x(x-l)/l^2}{1 + (2u/\pi)^2S}\right] \qquad (32)$$

and for the clamped case is given by

$$w(x) = \frac{ql^4[1 + (2u/\pi)^2S]^2}{16EIu^3}[\sin kx + \cot u(\cos kx - 1)] + \frac{ql^2[1 + (2u/\pi)^2S]}{8EIu^2}x(x-l) \qquad (33)$$

where $u = kl/2$ and $S = (E/G^*)(h/l)^2$.

Calculating the maximum nondimensionalized deflection for both cases it is seen that

$$\frac{32EI}{ql^4}w\left(\frac{l}{2}\right) = \frac{1 + (2u/\pi)^2S}{u^4}\left\{2\left[1 + \left(\frac{2u}{\pi}\right)^2 S\right]\cdot[\sec u - 1] - u^2\right\}$$

(simply supported beam column) (34)

and

$$\frac{32EI}{ql^4}w\left(\frac{l}{2}\right) = \frac{1 + (2u/\pi)^2S}{u^2}\left\{\frac{2}{u}\left(\frac{1 - \cos u}{\sin u}\right)\cdot\left[1 + \left(\frac{2u}{\pi}\right)^2 S\right] - 1\right\}$$

(clamped beam column) (35)

Allowing the values of $u$ to correspond to 10, 50, 80, and 90% of the buckling load** the results are seen in Fig. 1 and Fig. 2. Once again the clamped case is seen to be the most sensitive to increasing $S$.

### Comments and Conclusion

In closing two comments are offered concerning the analysis as well as a brief conclusion. 1) The answers obtained for the deflections were compared with the $S^* = 0$ case (classical beam theory) since the designer usually uses classical beam theory in his structural analysis procedures (and does *not* usually use isotropic Timoshenko beam theory, i.e., shear deformation theory with $E/G^* \cong 2.6$); and 2) The $E/G^*$ used in arriving at

§ The other Green's functions $C^{\delta M}$ and $C^{\psi M}$ are easily calculated but are of less interest and are hence omitted.

† $S^* = 0.1$ is approximately equivalent to $E/G^* = 50$ and $h/l = 1/7$.

** For the simply supported beam the value of $u$ at buckling is $\pi/2$ and for a clamped beam it is $\pi$.

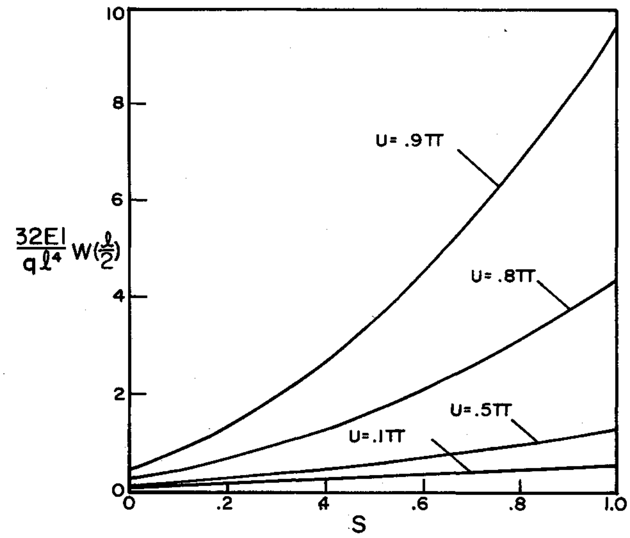


**Fig. 1 Nondimensional deflection at $x = l/2$ for a simply supported beam column.**

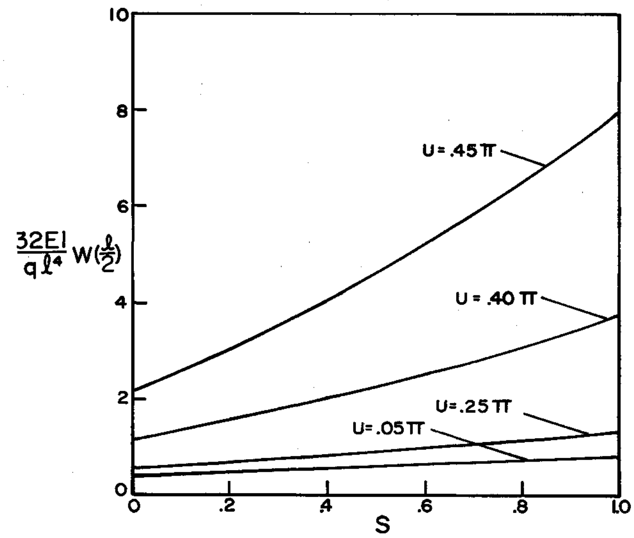


**Fig. 2 Nondimensional deflection at $x = l/2$ for a clamped beam column.**

$S^* \cong 0.10$, that is $E/G^* = 50$, is representative of some forms of pyrolytic graphite. However other new transversely isotropic materials may have an $E/G^*$ ratio of 200 or larger,†† thus it is quite possible to talk meaningfully about a value of $S^* = 0.20$ or larger.‡‡

In conclusion, the effect of transverse isotropy (and beam geometry) as embodied in $S$ (or $S^*$) is seen to significantly alter the solutions of the static problems solved. These effects are deleterious and hence must be considered when designing structures that utilize transversely isotropic materials. Finally, it is noted that these deleterious effects become more pronounced as the boundary restraints become more severe.

## References


[1] Wu, C. I. and Vinson, J. R., "On the Free Vibrations of Plates and Beams of Pyrolytic Graphite Type Materials," *AIAA Journal*, Vol. 8, No. 2, Feb. 1970, pp. 246–251.

[2] Brunelle, E. J., "The Statics and Dynamics of a Transversely Isotropic Timoshenko Beam," *Journal of Composite Materials*, Vol. 4, 1970, pp. 404–416.


†† This was learned in a recent conversation with the editor of *The Journal of Composite Materials*, S. Tsai.

‡‡ For example, if $E/G^* = 200$ and $h/l = 1/10$ $S^* \cong 0.20$.


[3] Brunelle, E. J., "The Elastic Stability of a Transversely Isotropic Timoshenko Beam," *AIAA Journal*, Vol. 8, No. 12, Dec. 1970, pp. 2271–2273.

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# Compressible Boundary-Layer Equations Solved by the Method of Parametric Differentiation


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### Nomenclature

$f$ = nondimensional stream function (as defined in Ref. 6)
$g$ = $\partial f/\partial\beta$
$g_1$ = nonhomogeneous part of $g$ defined in Eq. (7)
$g_2$ = homogeneous part of $g$ defined in Eq. (7)
$S$ = nondimensional enthalpy function (as defined in Ref. 6)
$T$ = $\partial S/\partial\beta$
$T_1$ = nonhomogeneous part of $T$ defined in Eq. (8)
$T_2$ = homogenous part of $T$ defined in Eq. (8)
$\beta$ = the pressure gradient parameter
$\eta$ = the similarity variable
$\lambda$ = $g''(0)$
$\mu$ = $T'(0)$
*Subscripts*
$w$ = wall or surface value

### Introduction

The method of parametric differentiation was developed by Rubbert[1] and Rubbert and Landahl[2] in connection with solving the transonic flow problems. This was a slight generalization of the concept of infinitesimal perturbation around a known solution originally given by Landahl.[3,4] Rubbert and Landahl[5] have applied this method for solving Falkner-Skan equations.

In this Note we illustrate this method by way of solving the compressible boundary-layer equations for Prandtl number unity. In the present problem $\beta$ is the parameter for different values of which the solution is sought.

### Equations and Solutions

Assuming such pressure gradients which give similarity solutions and the Prandtl number unity the boundary layer equations governing the flow of a compressible fluid become (Ref. 6, p 69)

$$f''' + ff'' + \beta(S - f'^2) = 0; \quad S'' + fS' = 0 \qquad (1)$$

The boundary conditions are

$$f(0) = f'(0) = 0, \quad f(\infty) = 1 \qquad (2a)$$

$$S(0) = S_w, \qquad S(\infty) = 1 \qquad (2b)$$

When $\beta \neq 0$ the Eqs. (1) together with the conditions (2) are a set of coupled equations required to be solved simultaneously. In the application of the present method the first step is to differentiate the Eqs. (1) and the boundary conditions (2) with respect to $\beta$ to obtain

$$g''' + fg'' + f''g + \beta(T - 2f'g') + (S - f'^2) = 0 \qquad (3a)$$

$$T'' + fT'' + gS' = 0 \qquad (3b)$$

$$g(0) = g'(0) = g'(\infty) = T(0) = T(\infty) = 0 \qquad (4)$$


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‡ Primes denote differentiation with respect to $\eta$.